# SEQUENCES

**INFINITE SEQUENCE:** Infinite sequence of real numbers is a function $a: N \to R$ and it is denoted by $\{a_n\}$ where $a_n = a(n)$, $n \in N$.

Ex1: $a_n = \frac{1}{n}$.

$\{a_n\} = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (n \in N)$

$a_1 \quad a_2 \quad a_3 \quad a_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Ex2: $b_n = (-1)^{n+1}$

$\{b_n\} = (-1)^{n+1} = 1, -1, 1, -1, \ldots\ldots\ldots\ldots\ldots\ldots (n \in N)$

$b_1 \quad b_2 \quad b_3 \quad b_4 \ldots\ldots\ldots\ldots\ldots\ldots$

Ex3: $c_n = K$.

$\{c_n\} = K = K, K, K, K, K \ldots\ldots\ldots\ldots\ldots\ldots\ldots (n \in N)$

$c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

It is called a ***"Constant Sequence"*** where K is an arbitrary constant.

Sequences are often defined recursively.

Ex: $a_1 = 1$ and $a_n = a_{n-1} + 1$ then

$a_2 = a_1 + 1 = 1+1=2$

$a_3 = a_2 + 1 = 2+1=3$

The sequence is 1, 2, 3,.................................

## Convergence and divergence:

The sequence of real numbers $\{a_n\}$ converges to a number L if for every $\in > 0$ there exists a natural number N such that

$$| a_n - L | < \in \text{ for all } n \geq N.$$

We say that $\{a_n\}$ diverges if no such number L exists.

If $\{a_n\}$ converges to L then we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ and call L the limit of the sequence $\{a_n\}$.

A sequence $\{ a_n \}$ ***diverges to infinity*** if for every number M there is a natural number N such that $a_n > M$ for all $n \geq N$. this holds then write $\lim_{n\to\infty} \mathbf{a_n} = \infty$ or $a_n \to \infty$ as $n \to \infty$.

A sequence $\{ a_n \}$ ***diverges to negative infinity*** if for every number m there is a natural number N such that $a_n < m$ for all $n \geq N$. this holds then write $\lim_{n\to\infty} \mathbf{a_n} = -\infty$ or $a_n \to -\infty$ as $n \to \infty$.

Ex:

1) $\{a_n\} = \frac{1}{n}$ *is a sequence* .

   If n=1 $\quad a_1 = 1$

   If n=2 $\quad a_2 = ½$

   If n=3 $\quad a_3 = 1/3$

   If $n \to \infty \quad a_n = \frac{1}{\infty} = 0.$

   i.e, $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots 0.$ ( as $n \to \infty$ , $\frac{1}{n} \to 0$.)

   Therefore $\{\frac{1}{n}\}$ is a convergent sequence and it converges to 0.

2) $\{b_n\} = \sqrt{n}$ is a sequence.

If n=1 $\quad b_1 = \sqrt{1}$

If n=2 $\quad b_2 = \sqrt{2}$

If n=3 $\quad b_3 = \sqrt{3}$.

If $n \to \infty \quad b_n = \sqrt{\infty} = \infty$

i.e, $\{\sqrt{n}\} = 1, \sqrt{2}, \sqrt{3}, \dots \dots \dots \dots \dots . \infty$ ( as $n \to \infty$ , $\sqrt{n} \to \infty$ )

Therefore $\{\sqrt{n}\}$ is a divergent sequence and it diverges to $\infty$.

3) $\{c_n\} = -n^2$ is a sequence.

If n=1 $\quad c_1 = -1$

If n=2 $\quad c_2 = -4$

If n=3 $\quad c_3 = -9$

If $n \to \infty \quad c_n = -(\infty)^2 = -\infty$

i.e, $\{-n^2\} = -1, -4, -9, \dots\dots\dots\dots - \infty$ ( as $n \to \infty$, $\{-n^2\} \to -\infty$.)

$\{-n^2\}$ is a divergent sequence and it diverges to $-\infty$.

4) $\{d_n\} = (-1)^{n+1}$ is a sequence.

If n=1 $\quad d_1 = +1$

If n=2 $\quad d_2 = -1$

If n=3 $\quad d_3 = +1$

If n=4 $\quad d_4 = -1$

as $n \to \infty$ we can't say that $d_\infty = 1$ or $d_\infty = -1$

i.e, $\{(-1)^{n+1}\} = 1, -1, 1, -1, \dots\dots\dots\dots$

Therefore $(-1)^{n+1}$ is a divergent sequence and it oscillates finitely between -1 and 1.

5) $\{e_n\} = (-1)^{n+1}\ n^2$

If n=1 $e_1 = 1$

If n=2 $e_2 = -4$

If n=3 $e_3 = 9$

If n=4 $e_4 = -16$

$\{e_n\} = (-1)^{n+1} n^2 = 1, -4, 9, -16, \ldots\ldots\ldots\ldots\ldots\ldots$

It is a divergent sequence and oscillates infinitely.

| | |
|---|---|
| **Convergent Sequence** | Converges to a Finite Value ( Ex. 1 ) |
| **Divergent Sequence** | may diverge to $\infty$ ( Ex. 2 ) |
| | may diverge to $-\infty$ ( Ex. 3 ) |
| | Oscillates finitely ( Ex. 4 ) |
| | Oscillates infinitely ( Ex. 5 ) |

## Boundedness of a sequence:

A sequence $\{a_n\}$ of real numbers is ***bounded above*** if there exists a number M such that $a_n \leq M$ $\forall$n. The number M is an upper bound for $\{a_n\}$.

A sequence $\{a_n\}$ of real numbers is ***bounded below*** if there exists a number m such that $m \leq a_n$ $\forall$n. The number m is an lower bound for $\{a_n\}$.

A sequence $\{a_n\}$ of real numbers is ***bounded*** if it is both ***bounded above and bounded below***, i.e, if there exist numbers m and M such that $m \leq a_n \leq M$ $\forall$n.

A number M is the ***least upper bound(lub or supremum )*** for $\{a_n\}$ if M is an upper bound for $\{a_n\}$ and no number less than M is an upper bound for $\{a_n\}$.

A number m is the ***greatest lower bound(glb or infimum)*** for $\{a_n\}$ if m is a lower bound for $\{a_n\}$ and no number greater than m is a lower bound for $\{a_n\}$.

Ex1:

$$\{a_n\} = \frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots\ldots .1 \textit{ is a bounded sequence.}$$

$$\underbrace{\ldots\ldots\ -3\ \ -2\ \ -1\ \ 0}_{\textit{Lower bounds}}\ \overset{\textbf{Infimum}}{\tfrac{1}{2}}\ \underbrace{2/3\ \ 3/4\ \ 4/5,\ldots\ldots\ldots}_{\textit{Sequence}}\ \overset{\textbf{Supremum}}{1}\ \underbrace{2\ \ 3\ \ 4\ \ 5\ \ldots\ldots}_{\textit{Upper bounds}}$$

Ex2:

$$\{a_n\} = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots\ldots .0 \textit{ is a bounded sequence.}$$

$$\underbrace{\ldots\ldots 6\ \ 5\ \ 4\ \ 3\ \ 2}_{\textit{Upper bounds}}\ \overset{\textbf{Supremum}}{1}\ \underbrace{\tfrac{1}{2}\ \ \tfrac{1}{3}\ \ \tfrac{1}{4}\ \ \ldots\ldots\ldots}_{\textit{Sequence}}\ \overset{\textbf{Infimum}}{0}\ \underbrace{-1\ -2\ -3\ -4\ \ldots\ldots}_{\textit{Lower bounds}}$$

Ex3:

$\{a_n\}$ = n= 1, 2, 3, 4, 5 ..................$\infty$

$$\underbrace{\ldots\ldots\ldots -3\ \ -2\ \ -1\ \ 0}_{\textit{Lower bounds}}\ \overset{\textbf{infimum}}{1}\ \underbrace{2\ \ 3\ \ 4\ \ 5\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty}_{\textit{Sequence}}$$

The above sequence is bounded below by 1 but not bounded above and no supremum. Therefore it is not a bounded sequence.

Ex4:

$\{a_n\} = -n = -1, -2, -3, -4$ .................. $-\infty$

***Supremum***

..........-3 -2 **-1** 0 1 2 3 4 5 ..........................................$\infty$

*Sequence* *Upper bounds*

The above sequence is bounded above by -1 but not bounded below and no infimum. Therefore it is not a bounded sequence.

Ex5:

$\{a_n\} = (-1)^{n+1} = 1, -1, 1, -1, 1, -1$...... a divergent sequence.

***Infimum*** ***Supremum***

...... -3 -2 **-1** **1** 2 3 4 5 ...........................$\infty$

*Lower bounds* *Sequence* *Upper bounds*

The above sequence is bounded below by -1 and bounded above by 1. Therefore it is a bounded sequence. ( But it is a divergent sequence.)

***Every convergent sequence is bounded but bounded sequence need not be convergent.***

Note: ***Infimum*** (m) ≤ ***Supremum*** (M) in any sequence.

## Non-decreasing and Non- increasing sequence:

Let $\{a_n\}$ be a sequence of real numbers. $\{a_n\}$ is said to be a non - decreasing sequence if $a_n \leq a_{n+1}$ for all n ϵ N.

$\{a_n\}$ is said to be a non -increasing sequence if $a_n \geq a_{n+1}$ for all n ϵ N.

*A sequence of real numbers is said to be a Monotonic sequence if it is either Non-decreasing or non-increasing.*

Ex:

$\{\frac{n}{n+1}\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots\ldots\ldots .. \textit{is a non} - \textit{decreasing sequence}.$

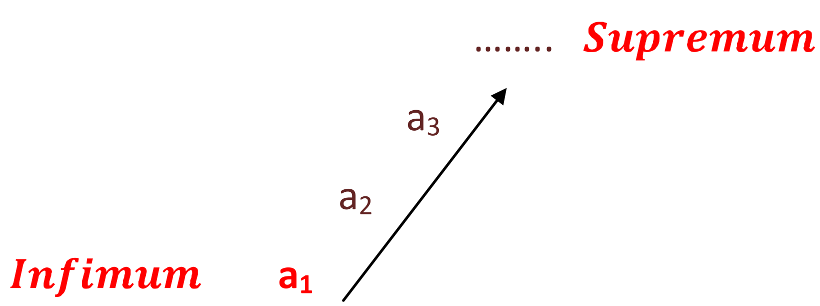
$\{\frac{n+1}{n}\} = 2, \frac{3}{2}, \frac{4}{3}, \ldots\ldots\ldots\ldots\ldots .. \textit{is a non} - \textit{increasing sequence}.$

**{K} = K, K, K,…………………… is a non-decreasing and non -increasing sequence.**

***Note: Every constant sequence is both non-increasing and non-decreasing.***

## Non-decreasing sequence:

Let $\{a_n\}$ be a non-decreasing sequence. ( $a_1 < a_2 < a_3$ ……….. )




1. It is bounded below by $\mathbf{a_1}$ (first term of the sequence) and g.l.b. or Infimum is $a_1$.
2. ***If it is bounded above, then it is bounded and then it converges to it's l.u.b. or supremum.***
3. If non-decreasing sequence is not bounded above then it has no supremum and it diverges to $\infty$.

## Non-increasing sequence:

Let $\{a_n\}$ be a non-increasing sequence. ( $a_1 > a_2 > a_3$ ………. )

***Supremum*** $a_1$

$a_2$

$a_3$

……. ***Infimum***

1. It is bounded above by $a_1$ ( first term of the sequence ) and l.u.b. or supremum is $a_1$.
2. ***If it is bounded below, then it is bounded and converges to its g.l.b. or infimum.***
3. If non-increasing sequence is not bounded below, then it diverges to $-\infty$.

***A monotonic sequence of real numbers is convergent if and only if it is bounded.***

## How to check the given sequence is convergent or not?

**Apply limit to the given sequence.**

1. **If it is a finite value L , then the sequence converges to L.**
2. **If it is not a finite value or limit does not exists, then the given sequence diverges.**

**Ex1:** $\{a_n\} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0.$ **( finite value )**

**Therefore the sequence $\{\frac{1}{n}\}$ converges to 0.**

Ex2: $\{a_n\} = \frac{n+1}{n}$

$\lim_{n\to\infty} \frac{n+1}{n}$ ( $\frac{\infty}{\infty}$ form )

= $\lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n}$ ( L' Hopital's Rule )

= $\lim_{n\to\infty}(1 + \frac{1}{n})$ = 1 ( finite value )

Therefore the sequence { $\frac{n+1}{n}$ } converges to 1.

Ex3: $\{a_n\} = \frac{3n^2+2n}{5n+1}$

$\lim_{n\to\infty} \frac{n(3n+2)}{n(5+\frac{1}{n})}$ ( $\frac{\infty}{\infty}$ form ) then $\lim_{n\to\infty} \frac{(3n+2)}{(5+\frac{1}{n})} = \infty$

Therefore the sequence { $\frac{3n^2+2n}{5n+1}$ } diverges to $\infty$.

Ex4: $\{a_n\} = \frac{n}{\sqrt{n+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n+1}}$ = $\lim_{n\to\infty} \frac{n}{\sqrt{n}(1+\frac{1}{\sqrt{n}})}$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{(1+\frac{1}{\sqrt{n}})} = \infty$$

Therefore the sequence { $\frac{n}{\sqrt{n+1}}$ } diverges to $\infty$.

Ex 5: $\{a_n\} = (-1)^{n+1}$

$\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

Therefore the sequence { $(-1)^{n+1}$ } is a divergent sequence.

**Ex 6:** $\{a_n\} = \sqrt{n+1} - \sqrt{n}$

First we have to rationalize the given sequence.

$$\sqrt{n+1} - \sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ ( finite value)

Therefore the sequence $\{\sqrt{n+1} - \sqrt{n}\}$ converges to 0.

## Remember the following formulae:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$ where $x > 0$
4. $\lim_{n\to\infty} x^n = 0$ where $|x| < 1$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any x
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any x

## Subsequence:

If $\{a_n\}$ is a sequence and $\{n_k\}$ is a sequence of natural numbers such that $n_1 < n_2 < n_3 <$ ……. Then the sequence $\{a_{n_k}\}$ is called a Subsequence of $\{a_n\}$.

Note:

1. The sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

2. If subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ of a sequence $\{a_n\}$ converge to the same limit L, Then the sequence $\{a_n\}$ converges to L.
   Ex: $\{a_n\} = \frac{(-1)^{n+1}}{n^2} = 1, \frac{-1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \frac{1}{49}, -\frac{1}{64}$,…………
   $\{a_{2n-1}\} = 1, \frac{1}{9}, \frac{1}{25}, \frac{1}{49}$,…………………… converges to 0
   $\{a_{2n}\} = -\frac{1}{4}, -\frac{1}{16}, -\frac{1}{36}$, ……………………… converges to 0.
   The subsequences are converging to 0. Therefore the given sequence $\{a_n\} = \frac{(-1)^{n+1}}{n^2}$ also converges to 0.

3. If subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ of a sequence $\{a_n\}$ converge to different limits $L_1 \neq L_2$, Then the sequence $\{a_n\}$ diverges.
   Ex: $\{a_n\} = (-1)^{n+1}$ = 1, -1, 1, -1, 1, -1,…………………..
   $\{a_{2n-1}\}$ = 1, 1, 1, 1, …………………………. Converges to 1.
   $\{a_{2n}\}$ = -1, -1, -1, -1,…………………………converges to -1.
   The subsequences are converging to two different limits $1 \neq -1$.
   Therefore the given sequence $\{a_n\} = (-1)^{n+1}$ is a divergent sequence.

## The sandwich theorem for Sequences:

Let $\{a_n\}$ , $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers and $a_n \leq b_n \leq c_n$ for all n beyond some index N. if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Ex: $\{a_n\} = \frac{\cos n}{n^2}$

We know that $-1 \leq \cos n \leq 1$ for all n.

$$\frac{-1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}$$

$\lim_{n\to\infty} \frac{-1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$ then by Sandwich theorem $\lim_{n\to\infty} \frac{\cos n}{n^2} = 0$

Therefore the given sequence $\{a_n\} = \frac{\cos n}{n^2}$ converges to 0.

**It is not the matter, how many number of breathes we have been taking. It is the matter how many breathless moments we are passing through.**

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