SERIES

INFINITE SERIES: An Infinite Series of real numbers is the sum of an infinite sequence of real numbers.

If $\{a_n\}$ is an infinite sequence of real numbers, then the expression $a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots\ldots + a_n + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ is called an Infinite series denoted by $\sum_{n=1}^{\infty} a_n$.

a_n is called the n^{th} term of the Infinite Series.

Ex1:

$\{a_n\} = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$is an infinite sequence. then

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$is an infinite Series formed by the above infinite sequence.

Ex2:

$\{a_n\} = (-1)^{n+1} = 1, -1, 1, -1, 1, -1, \ldots\ldots$is an infinite sequence and $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 + (-1) + 1 + (-1) + \ldots\ldots\ldots$ is an infinite series formed by the above infinite sequence.

Convergence and Divergence of a Series:

If the sum of an infinite series is a finite value L, then we say that the series converges to L. if the sum of an infinite series is not a finite value, then we say that the series diverges.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots$ Is an infinite series.

$S_1 = 1$

$S_2 = 1 + \frac{1}{2^2}$

$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}$

....

$\mathbf{S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots \frac{1}{n^2}}$ is called the ***n^{th} partial sum*** of the Series.

Methods to Check the Convergence of Series:

The Series $\sum_{n=1}^{\infty} a_n$ converges if the Sequence of n^{th} partial sums $\{ s_n \}$ Converges.

Ex:

$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ is an infinite series.

n^{th} partial sum $S_n = \sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right)$

$= \sum_{k=1}^{n} [\ln k - \ln(k+1)]$

$= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \ldots\ldots\ldots\ldots (\ln n - \ln(n+1))$

$= \ln 1 - \ln(n+1)$

$S_n = -\ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty$

Thus, the sequence of n^{th} partial sums of the series diverges. Therefore the given series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges.

Geometric Series test:

If the given series is in the form

$\sum_{n=1}^{\infty} ar^{n-1}$ = a + ar + ar^2 + ……….+ ar^{n-1}+……….

In which a and r are the fixed real numbers and a ≠ 0. Then the series is called a " Geometric Series ".

The Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^{n}$

1. Converges to $\frac{a}{1-r}$ if |r| < 1.
2. Diverges if |r| ≥ 1.

n^{th} Term test:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$ or fails to exists.

Note: if $\lim_{n\to\infty} a_n = 0$, then we cannot say that the series converges.

1. If $\sum a_n$ = A and $\sum b_n$ = B are convergent Series, then $\sum(a_n + b_n)$ = A + B and $\sum(a_n - b_n)$ = A - B and $\sum k\, a_n$ = kA.
   That means, the Sum and Difference of two convergent series are also convergent and non zero constant multiple of a convergent series is also convergent.
2. Every non – zero constant multiple of a divergent series is divergent.
3. If one of the series $\sum a_n$ and $\sum b_n$ coneverges and the other diverges then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ diverge.
4. If $\sum a_n$ and $\sum b_n$ both divergent series, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ can converge.
   Ex: $\sum a_n$ = 1 + 1 + 1 + 1 + ……………………………. diverges to ∞
   $\sum b_n$ = -1 + -1 + -1 + -1 + …………………….. divereges to -∞
   $\sum(a_n + b_n)$ = (1 -1) + (1-1) + (1-1) + ………

$= 0 + 0 + 0 + 0 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ converges to 0.

Here, $\sum a_n$ and $\sum b_n$ both divergent series but $\sum(a_n + b_n)$ convergent series.

Note: Addition or deletion of a finite number of terms from a series will not alter its convergence or divergence.

Integral Test:

Let $\{ a_n \}$ be a sequence of positive terms. Suppose $a_n = f(n)$, where f is continuous, positive valued decreasing function of x for $x \geq N$, where N is a natural number.

Then the series $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

P – Series test:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots\ldots\ldots$ Where P is a real constant, converges if P > 1 and diverges if P ≤ 1.

Ex: $\sum \frac{1}{n}$, $\sum \frac{1}{\sqrt{n}}$ are divergent series.

$\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$, $\sum \frac{1}{n^4}$, …… are convergent series.

Logarithmic P – Series test:

The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} = \frac{1}{2(\ln 2)^p} + \frac{1}{3(\ln 3)^p} + \frac{1}{4(\ln 4)^p} + \ldots\ldots\ldots\ldots$ where p is a real constant, converges if p > 1 and diverges if p ≤ 1.

Comparison test:

Let $\sum a_n$ be a series with non – negative terms.

1. $\sum a_n$ converges if and only if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n \geq N$, for some natural number N.
2. $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n$ for all $n \geq N$, for some natural number N.

The Limit comparison Test:

$\sum a_n$ and $\sum b_n$ be series and $a_n > 0$, $b_n > 0$, for all $n \geq N$, for some natural number N.

1. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$ **then** $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Note:

Choose $\sum b_n$ as a geometric series like $\sum \frac{1}{2^n}$, $\sum \frac{1}{3^n}$, $\sum \frac{1}{4^n}$, ……………. Or P – series like $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$, $\sum \frac{1}{n^4}$, …….etc. for the limit comparison test. (numerator of $\sum b_n$ should be 1)

The Ratio Test:

Let $\sum a_n$ be a series with positive terms and suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$. then

1. The series converges if $l < 1$.
2. The series diverges if $l > 1$ or l is infinite.
3. The test is inconclusive if $l = 1$.

Note:

The Ratio test is effective when the terms of a series contain factorial expressions involving n or expressions raised to a power of n.For example $\sum \frac{2^n}{n!}$, $\sum \frac{2^n+5}{3^n}$, $\sum \frac{4^n\,(n!)^2}{(2n!)}$ etc.

The Root Test:

$Let\ \sum a_n$ be a series with $a_n \geq 0$ for $n \geq M$, for some natural number M and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = l$. then

1. The series converges if $l < 1$.
2. The series diverges if $l > 1$ or l is infinite.
3. The test is inconclusive if $l = 1$.

Alternating Series:

A series in which the terms are alternately positive and negative is called an " Alternating Series ".

Ex: $\sum \frac{(-1)^{n+1}}{n}$, $\sum \frac{(-1)^n\ 4}{2^n}$, etc.

The Alternating Series Test: (Leibniz's Theorem)

The alternating series $\sum(-1)^{n+1}\, a_n = a_1 - a_2 + a_3 - a_4 + \ldots\ldots$.converges if

1. $a_n > 0$ for all n ∈ N
2. $a_n \geq a_{n+1}$ for all n ∈ N
3. $\lim_{n\to\infty} a_n = 0$.

Ex: The alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ convergent.

Absolute Convergence:

A series $\sum a_n$ converges Absolutely if the corresponding series of absolute values, $\sum |a_n|$ converges.

Note: Every absolutely convergent series is convergent. Means if $\sum |a_n|$ converges then $\sum a_n$ also converges.

Conditional Convergence:

A series converges conditionally if it converges but does not converges absolutely. that means $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ex: The alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges conditionally.

$\sum a_n$	$\sum \lvert a_n \rvert$	**Then the series is**
$\sum a_n$ convergent	**$\sum \lvert a_n \rvert$ convergent**	**Absolutely Convergent**
$\sum a_n$ convergent	**$\sum \lvert a_n \rvert$ divergent**	**Conditionally Convergent**
$\sum a_n$ divergent	**$\sum \lvert a_n \rvert$ divergent**	

Note:

1. ***If $\sum |a_n|$ converges then $\sum a_n$ converges.***
2. ***If $\sum a_n$ diverges then $\sum |a_n|$ diverges.***

The alternating P – Series Test:

The alternating p – series $\sum \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots\ldots\ldots\ldots$

1. Converges if $p > 0$
2. Converges absolutely if $p > 1$
3. Converges conditionally if $0 < p \leq 1$.

The Ratio Test:

Let $\sum a_n$ be a series of real numbers with an ≠ 0 , for all n and suppose that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = l$. then

1. The series converges absolutely if $l < 1$.
2. The series diverges if $l > 1$ or l is infinite.
3. The test is inconclusive if $l = 1$.

The Root Test:

Let $\sum a_n$ be a series of real numbers and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l$. then

1. The series converges absolutely if $l < 1$.
2. The series diverges if $l > 1$ or l is infinite.
3. The test is inconclusive if $l = 1$.

The Power Series:

Let a be given real number and x be a real variable. A power series in $x - a$ or a power series centered at a or a power series about a is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \ldots\ldots a_n (x-a)^n + \ldots..$$

Where a_n's are constants called coefficients of the series.

Note:

1. The power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ may converges at exactly at x = a.

2. The power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ may converges in some interval with radius R where a is the centre of that interval.
3. The power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ may converges for the values of x at everywhere on the real line.

Radius of Convergence:

1. **If a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges in some interval (p,q) where 'a' is the centre, then the radius of the convergence is the half of the distance from p to q on the real line.**
2. **If a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges exactly at a, then the radius of convergence is zero.**
3. **If a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges for the values of x at everywhere on the real line, then the radius of the convergence is ∞.**

Term by Term Differentiation:

The power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges to Some function $f(x)$ in some interval $a-R<x<a+R$, we can write it as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad \text{where } a-R<x<a+R$$

By the term by term Differentiation theorem,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad \text{where } a-R<x<a+R$$

$$f^{|}(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1} \qquad \text{where } a-R<x<a+R$$

$$f^{||}(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x-a)^{n-2} \qquad \text{where } a-R<x<a+R$$

And so on. It means that the derivatives of the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ are $\sum_{n=0}^{\infty} n a_n (x-a)^{n-1}$, $\sum_{n=0}^{\infty} n(n-1) a_n (x-a)^{n-2}$

……… are also convergent series and converge to $f^{|}(x)$, $f^{||}(x)$, ……… respectively in the same interval $a - R < x < a + R$.

Term by Term Integration:

The power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges to Some function $f(x)$ in some interval $a - R < x < a + R$, we can write it as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad \text{where } a - R < x < a + R$$

By the term by term Integration theorem,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad \text{where } a - R < x < a + R$$

$$\int f(x)\, dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C \qquad \text{where } a - R < x < a + R$$

$$\int\left(\int f(x)\, dx\right) dx = \sum_{n=0}^{\infty} a_n \frac{1}{n+1} \frac{(x-a)^{n+2}}{n+2} + C \text{ where } a - R < x < a + R$$

And so on. It means that the integrals of the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$are $\sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C$, $\sum_{n=0}^{\infty} a_n \frac{1}{n+1} \frac{(x-a)^{n+2}}{n+2} + C$ ………are also convergent series and converge to $\int f(x)\, dx$, $\int\left(\int f(x)\, dx\right) dx$ ……… respectively in the same interval $a - R < x < a + R$.

Taylor Series:

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor Series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^k(a)}{k!} (x-a)^k = f(a) + f^{|}(a)(x-a) + \frac{f^{||}(a)}{2!} (x-a)^2 + \ldots\ldots\ldots\ldots \frac{f^n(a)}{n!} (x-a)^n + \ldots\ldots\ldots\ldots$$

By Using the Taylor series formula, we can find a power series to a given function $f(x)$ by at given real number a.

Maclaurin Series:

If we take the real number $a = 0$ in the above Taylor Series, then the taylor series becomes as

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!}(x)^{k} = f(a) + f^{|}(0)(x) + \frac{f^{||}(0)}{2!}(x)^{2} + \ldots\ldots\frac{f^{n}(0)}{n!}(x)^{n} + \ldots\ldots\ldots$$

It is called a " Maclaurin Series".

Taylor Polynomial:

From the Taylor Series Formula,

$$\sum_{k=0}^{\infty} \frac{f^{k}(a)}{k!}(x-a)^{k} = f(a) + f^{|}(a)(x-a) + \frac{f^{||}(a)}{2!}(x-a)^{2} + \ldots\ldots\ldots\ldots \; . \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\frac{f^{n}(a)}{n!}(x-a)^{n} + \ldots\ldots\ldots\ldots\ldots$$

We can write the Taylor Polynomials as

Polynomial of Order 1 is $P_1 = f(a) + f^{|}(a)(x-a)$

Polynomial of Order 2 is $P_2 = f(a) + f^{|}(a)(x-a) + \frac{f^{||}(a)}{2!}(x-a)^{2}$

Polynomial of Order 3 is $P_3 = f(a) + f^{|}(a)(x-a) + \frac{f^{||}(a)}{2!}(x-a)^{2} + \frac{f^{|||}(a)}{3!}(x-a)^{3}$

General form of Taylor polynomial of Order n is

$$\boldsymbol{P_n = f(a) + f^{|}(a)(x-a) + \frac{f^{||}(a)}{2!}(x-a)^{2} + \ldots\ldots\ldots\ldots\ldots \frac{f^{n}(a)}{n!}(x-a)^{n}} \; .$$

Note:

We can use these Taylor polynomials to get approximate value of the function $f(x)$ at given real number a.

Taylor Formula:

By mentioned in the above note, we can estimate the value of function $f(x)$ at given real number a by using the Taylor Polynomials. But the polynomials will not give the exact value of the function and some error R_n exists.

Suppose, if we are estimating the function $f(x)$ value at a by using the Taylor Polynomial of order 1 that is P_1 , then there will be some error R_1. We can write it as

$$f(x) = P_1 + R_1$$

Suppose, if we are estimating the function $f(x)$ value at a by using the Taylor Polynomial of order 2 that is P_2 , then there will be some error R_2. We can write it as

$$f(x) = P_2 + R_2$$

In the same way, we can write the general formula as

$$\boldsymbol{f(x) = P_n + R_n}$$

Where P_n is the Taylor Polynomial of Order n and R_n is the estimating error. The above formula is called the ***"Taylor Formula"*** .

In the Taylor Formula,

$$f(x) = P_n + R_n$$

If n → ∞ then $R_n \rightarrow 0$ (The estimating error will become Zero) and P_n will become the Taylor Series formula. (The Taylor Polynomial Order n will become the Taylor Series Formula as n → ∞).

As long as we have memories,
yesterday remains.
As long as we have hope,
tomorrow awaits.
As long as we have friendship,
each day is never a waste.

Jaya Krishna Reddy. M.